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RESEARCH

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Nonexpansive-type operators from nonlinear operator-valued measures

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Abstract

This paper is grounded in the long-standing, purely abstract theory of integration with respect to nonlinear operator-valued measures $\mu : \mathcal{R} \to N(S, F)$, which defines the integral operator $Tf \equiv \int f \, d\mu$ in full generality. By leveraging Pettis and Bochner Radon–Nikodým-type theorems and imposing natural assumptions on measurability and semivariation, we establish that this abstract integral can be represented by a classical scalar integral with a Carathéodory kernel $\varphi : S \times X \to F$. This representation, a core contribution of this work, translates the abstract operator integral into the more familiar form $Tf = \int_X \varphi(f(y), y) \, dm(y)$. This approach not only provides a powerful analytical framework but also enables the direct use of standard tools for nonlinear integral operators. Moreover, by introducing a Lipschitz condition on the kernel, we derive the nonexpansive inequality $||Tf - Tg|| \leq ||L||_{L^q} ||f - g||_{L^p}$, thereby classifying T as a nonexpansive operator (or a strict contraction when $||L||_{L^q} < 1$). Consequently, this framework enables the direct application of classical fixed-point algorithms, such as Picard iterations for contractions and Krasnoselskii–Mann or Halpern schemes for nonexpansive mappings. This work bridges the gap between the abstract theory of operator-valued measures and the practical application of nonexpansive-type operator theory and algorithms.

Keywords: Operator-valued measure; Integration with respect to nonlinear operator-valued measures; Pettis integral; Bochner integral; Radon-Nikodym representation; Carathéodory kernel; Nonexpansive operator; Krasnoselskii-Mann iteration; Halpern iteration; Fixpoint theory

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1 Introduction

The motivation for this paper stems from the desire to bridge two important fields of functional analysis: the abstract theory of nonlinear operator-valued measures and the practical application of fixed-point algorithms. While significant work has been done on integration with respect to linear operator-valued measures, much less attention has been given to the nonlinear case. This is a notable gap, especially given the growing interest in models described by nonlinear differential and integral equations, where key operators such as the Nemytskii and Urysohn operators arise naturally as integrals with respect to these measures (as shown in Examples 1.1 and 1.2 below).

This paper builds upon the concepts initiated in [48], where a general theory of integration for nonlinear operator-valued measures was developed. Our main contribution is to demonstrate that under certain conditions, this abstract integral can be represented by a classical integral with a Carathéodory kernel. This representation provides a powerful framework for studying the properties of integral operators and, crucially, for establishing conditions under which these operators are of the nonexpansive type. These conditions, in turn, allow for the direct application of standard iterative methods from fixed-point theory, such as Picard, Krasnoselskii-Mann, and Halpern iterations.

The subsequent sections of this paper are structured as follows. We begin by formally defining the key concepts and notations, many of which are drawn from [48] for consistency. We then present two core representation theorems: a Radon-Nikodym-type (RN) theorem for the Pettis integral in Sect. 2, and another one for the Bochner integral in Sect. 3. Finally, in Sect. 4, we leverage these representation theorems to show that the resulting integral operators are nonexpansive and can be analyzed using a variety of fixed-point algorithms, thereby closing the loop between the abstract theory and its practical algorithmic applications.

Let F be a Banach space and let S be a separable Banach space. By $N(S,F)$ we denote the space of all functions $U : S \to F$ satisfying two conditions: (1) $U(0) = 0$, and (2) U is uniformly continuous on every ball in S. Let further $\mathcal{R}$ be a σ-algebra of subsets of a non-empty set X.

Definition 1.1 A set function $\mu : \mathcal{R} \to N(S,F)$ will be called an operator-valued measure if it has the following properties:

1. $\mu(\emptyset) = 0$
2. $\mu(\bigcup_{i=1}^{\infty} B_i)r = \sum_{i=1}^{\infty} \mu(B_i)r$, for every $r \in S$ and for a sequence of pairwise disjoint sets (B_i) from $\mathcal{R}$.

It is important to note that an operator valued measure is countably additive in the strong operator topology (in the point-wise sense) but not necessarily countably additive in the uniform operator topology. By the Orlicz-Pettis theorem (see IV 10.1 in [16]), countable additivity in the strong and weak topologies is equivalent. Operator-valued measures and integration with respect to such measures have many applications, including scattering theory, spectral analysis, stochastic integration of vector-valued processes, operator equations, and the representation of operators in Banach function spaces. Several recent articles related to these topics have been published, see e.g., [21, 35], and the papers cited therein. We refer the reader to an excellent 2017 book by Jefferies [22] providing a monographic exposition of the current state of the art. These works are usually based on the classical concepts of integrals over operator-valued measures introduced in the papers of Bartle [1] and Dobrakov [13], which represent a special case of the theory discussed here when $N(S,F) = L(S,F)$, the space of all linear bounded operators. In this context, it seems interesting to develop further the concepts initiated in [48], concerning the situation when measures do not have to be linear operators. Even more so, because of the growing interest in scientific models described by nonlinear differential and integral, frequently stochastic, equations. Many important nonlinear operators like the Nemytskii operator and the Urysohn operator can be treated as integrals with respect to such measures. Nonlinear operator measures had also been considered in [24] in the context of applications of modular function spaces to the theory of nonlinear integral operators. Also, in 2015 in [23], Khamsi and Kozlowski refer to [48], presenting it as an important example of a modular function space in the context of fixed-point theory. The following definitions and notations are taken from [48]. The reader is referred to this paper for further details.

Definition 1.2 Given an additive set function $G : \mathcal{R} \to F$. For each $E \subset X$ the variation of G on a set E is denoted by $V(G)(E)$ and is defined by $V(G)(E) = \sup\{\sum_{i=1}^{n} \|G(E_i)\|\}$, where the supremum is taken over all finite sequences (E_i) of pairwise disjoint sets in $\mathcal{R}$ with $E_i \subset E$.

Definition 1.3 For every $\alpha > 0$ and every $E \subset X$ the semivariation of the operator-valued measure μ is defined by the formula $sv(\mu_\alpha, E) = \sup\{\| \sum_{i=1}^{n} [\mu(E_i)r_i]\|\}$, where the supremum is taken over all finite sequences (E_i) of pairwise disjoint sets in $\mathcal{R}$ with $E_i \subset E$ and over all $r_i \in S$ such that $\|r_i\| \leq \alpha$.

Definition 1.4 For every $\alpha, \delta > 0$ and every $E \subset X$

1. the δ-variation of the operator-valued measure μ is defined by the formula

$$\nu_\delta(\mu_\alpha, E) = \sup \left\{ \sum_{i=1}^{n} V(\mu(\cdot)r_i - \mu(\cdot)r_i')(E_i) \right\},$$

2. the δ-semivariation of the operator-valued measure μ is defined by the formula

$$s\nu_\delta(\mu_\alpha, E) = \sup \left\{ \left\| \sum_{i=1}^{n} \left[\mu(E_i)r_i - \mu(E_i)r_i'\right] \right\| \right\},$$

where the suprema are taken over all finite sequences (E_i) of pairwise disjoint sets in $\mathcal{R}$ with $E_i \subset E$ and over all pairs $r_i, r_i' \in S$ such that $\|r_i\| \leq \alpha$, $\|r_i'\| \leq \alpha$, $\|r_i - r_i'\| \leq \delta$.

It is obvious that $s\nu_\delta(\mu_\alpha, E) \leq \nu_\delta(\mu_\alpha, E)$ for every $E \in \mathcal{R}$.

Definition 1.5 We shall say that

1. the operator measure μ is locally uniformly continuous in variation on $E \in \mathcal{R}$ if $\lim_{\delta\to 0} \nu_\delta(\mu_\alpha, E) = 0$ for every $\alpha > 0$.
2. the operator measure μ is locally uniformly continuous in semivariation on $E \in \mathcal{R}$ if $\lim_{\delta\to 0} s\nu_\delta(\mu_\alpha, E) = 0$ for every $\alpha > 0$.

If μ is an operator-valued measure on $\mathcal{R}$, then, for each $r \in S$, $\|\mu(\cdot)r\|$ is a countably subadditive set function.

Definition 1.6 For every $r \in S$ and $E \subset X$ a submeasure majorant $\overline{\mu}$ for μ is defined by the formula $\overline{\mu}(E)r = \sup\{\|\mu(A)r\| : A \subset E, A \in \mathcal{R}\}$.

Definition 1.7 A set $E \in \mathcal{R}$ is called μ-null iff $\overline{\mu}(E)r = 0$ for every $r \in S$.

Assumption 1.1 The term μ-almost everywhere refers to the complement of a μ-null set. In this paper we assume that μ is complete, i.e., the σ-algebra $\mathcal{R}$ is such that every subset of a μ-null set is a member of $\mathcal{R}$.

Assumption 1.2 (Measure conditions) In this paper we assume that the operator measure μ meets the following conditions:

(μ1) for each $E \in \mathcal{R}$ and $\alpha > 0$ the δ-semivariation $sv_\delta(\mu_\alpha, E)$ tends to 0 as δ tends to 0, i.e. μ is locally uniformly continuous in semivariation on every $E \in \mathcal{R}$.

(μ2) for each $r \in S$ the submeasure majorant $\overline{\mu}(\cdot)r$ is continuous on $\mathcal{R}$, i.e., if $E_n \in \mathcal{R}$, $E_n \searrow \emptyset$, then $\lim_n \overline{\mu}(E_n)r = 0$.

Definition 1.8 Let m be a finite positive measure on the σ-algebra $\mathcal{R}$. We say that operator-valued measure μ is absolutely continuous with respect to measure m (what we write $\mu \ll m$) if for every $r \in S$, $\lim_{m(E)\to 0} \overline{\mu}(E)r = 0$.

Definition 1.9 A function $s : X \to S$ is called an $\mathcal{R}$-simple function on X with value in S if the function s is of the form $s = \sum_{i=1}^{k} r_i \cdot 1_{A_i}$, where (A_i) are pairwise disjoint sets in $\mathcal{R}$ and $\bigcup_{i=1}^{k} A_i = X$. By $\mathcal{E}$ we shall denote the space of all $\mathcal{R}$-simple functions.

Definition 1.10 A function $f : X \to S$ is called:

1. μ-measurable if there exists a sequence of $\mathcal{R}$-simple functions (s_n), such that $s_n \to f$ μ-almost everywhere on X.
2. m-measurable if there exists a sequence of $\mathcal{R}$-simple functions (s_n), such that $s_n \to f$ m-almost everywhere on X.

It is easy to check that in the case where $\mu \ll m$, a function f is μ-measurable if and only if it is m-measurable.

Definition 1.11 The average range of an additive set function $G : \mathcal{R} \to F$ on a measurable set $E \in \mathcal{R}$, $m(E) > 0$, is the set

$$A_E(G) = \left\{ \frac{G(E')}{m(E')} : E' \in \mathcal{R}, E' \subset E, m(E') > 0 \right\}.$$

A set function G has almost locally compact average range if whenever $E \in \mathcal{R}$, $m(E) > 0$ and $\epsilon > 0$ then there exists a measurable set E_o such that $m(E \setminus E_o) < \epsilon$ and $A_{E_o}(G)$ is a precompact set in F.

Definition 1.12 The definition of the integral with respect to the operator-valued measure was introduced in detail in [48].

1. For every $\mathcal{R}$-simple function s of the form $s = \sum_{i=1}^{k} r_i \cdot 1_{A_i}$ we define

$$\int_E s\, d\mu = \sum_{i=1}^{k} \mu(E \cap A_i) r_i, \quad \text{where } E \in \mathcal{R}, A_i \in \mathcal{R}.$$

2. A function $f : X \to S$ is called *totally* $\mathcal{R}$-measurable if there exists a sequence (s_n) of $\mathcal{R}$-simple functions converging uniformly to f. By $\mathcal{E}_0$ we shall denote the space of all *totally* $\mathcal{R}$-measurable functions.

3. For every *totally* $\mathcal{R}$-measurable function $f \in \mathcal{E}_0$ and $E \in \mathcal{R}$ we put $\int_E f\, d\mu = \lim_n \int_E s_n\, d\mu$, where (s_n) is a sequence of $\mathcal{R}$-simple functions converging uniformly to f.
4. We extend the domain of integration to the class $\mathcal{M}(\mu) = \{f : X \to S$ functions such that there exists a sequence (s_n) of $\mathcal{R}$-simple functions converging to f μ-almost everywhere and $\lim_n \int_E s_n\, d\mu$ exists for every $E \in \mathcal{R}\}$. Such functions are called μ-integrable functions.
5. For a μ-integrable function $f \in \mathcal{M}(\mu)$ and $E \in \mathcal{R}$ we put $\int_E f\, d\mu = \lim_n \int_E s_n\, d\mu$. The independence from the choice of sequence (s_n) was proved in [48].

Definition 1.13 By $M(\mu, m)$ we will denote the class of functions $f : X \to S$ such that there exists a sequence (s_n) of $\mathcal{R}$-simple functions which converges to f m-almost everywhere and that $\lim_n \int_E s_n\, d\mu$ exists for every $E \in \mathcal{R}$.

Remark 1.1 In an equivalent manner, $M(\mu, m)$ can be defined as the set of functions $f : X \to S$, such that f is m-measurable and there exists a non-decreasing sequence of sets (Z_k) such that $Z_k \in \mathcal{R}$, $(X \setminus Z_k) \searrow N$ (where N is m-null), $f \cdot 1_{Z_k}$ is a *totally* $\mathcal{R}$-measurable function and $\lim_k \int_E f \cdot 1_{Z_k}\, d\mu$ exists for every $E \in \mathcal{R}$. The proof of this fact can be made in the same way as in theorem 2.5 in [48].

Remark 1.2 It is obvious that if $\mu \ll m$ then $M(\mu, m) \subset \mathcal{M}(\mu)$. If in addition $m \ll \mu$, then $M(\mu, m) = \mathcal{M}(\mu)$.

Definition 1.14 Given $f \in \mathcal{M}(\mu)$ we define a set function $G_f : \mathcal{R} \to F$ by $G_f(A) = \int_A f\, d\mu$ for $A \in \mathcal{R}$. The space $MV(\mu, m)$ is defined by $MV(\mu, m) = \{f \in M(\mu, m) : V(G_f)(X) < +\infty\}$.

Definition 1.15 Let φ be a function $\varphi : S \times X \to F$.

1. φ is a Carathéodory function if $\varphi(r, \cdot)$ is a measurable function for each $r \in S$ and $\varphi(\cdot, x)$ is continuous for m-almost everywhere $x \in X$.
2. φ is a locally uniformly continuous Carathéodory function if $\varphi(r, \cdot)$ is a measurable function for each $r \in S$ and $\varphi(\cdot, x)$ is uniformly continuous on each ball in S for m-almost everywhere $x \in X$.
3. φ is a locally uniformly weakly continuous Carathéodory function if $\varphi(r, \cdot)$ is a measurable function for each $r \in S$ and the functional $< z^*, \varphi(\cdot, x) >$ is uniformly continuous on each ball in S for every $z^* \in F^*$ and for m-almost everywhere $x \in X$.

The following two examples show that some non-linear operators connected with the theory of integral equations can be represented as integrals with respect to some nonlinear operator-valued measures.

Example 1.1 Let $S = \mathbb{R}$, $F = L^1(X)$ and φ be a Carathéodory function, $\varphi(r, \cdot) \in L^1(X)$ for every $r \in \mathbb{R}$ and $\varphi(0, \cdot) \equiv 0$. Define μ as follows

$$(\mu(A)r)(x) := \varphi(r, x)\, \mathbf{1}_A(x), \ \ \textit{for}\ A \in \mathcal{R}.$$

For $f \in \mathcal{M}(\mu)$, the integral becomes

$$\Big(\int_X f\, d\mu\Big)(x) = \varphi\big(f(x), x\big),$$

i.e., the integral is equal to the Nemytskii operator.

Example 1.2 Let $(X, \mathcal{R}, m)$ be a measure space, $S = \mathbb{R}$, $F = L^1(X)$. Let a function $K : X \times X \times \mathbb{R} \to \mathbb{R}$ be such that

(1) $K(x, y, 0) \equiv 0$,
(2) $K(x, y, r)$ is continuous in $r \in \mathbb{R}$ almost everywhere in $X \times X$ and measurable in (x, y) for every fixed $r \in \mathbb{R}$,
(3) for every $\epsilon > 0$ there exists $\delta > 0$ such that $\int\limits_E |\int\limits_A K(x, y, r)\, dm(y)|\, dm(x) < \epsilon$ if $m(A) < \delta$.

Define $\mu : \mathcal{R} \to N(\mathbb{R}, L^1(X))$ by

$$(\mu(E)r)(x) = \int\limits_E K(x, y, r)\, dm(y) \in L^1(X), \quad x \in X.$$

For $f \in \mathcal{M}(\mu)$ we have

$$(\int\limits_E f\, d\mu)(x) = \int\limits_E K(x, y, f(y))\, dm(y), \quad x \in X.$$

Hence, the integral of f with respect to the nonlinear valued measure μ is the Urysohn operator. Assumptions (1)-(3) are standard in the theory of nonlinear integral equations.

The author's paper [48] proves a number of principal properties of integration in this sense. In particular, orthogonal additivity, absolute continuity of the integral, and also the Vitali-type theorem. The reader is referred to this paper for further details. Note that in [48] a more complicated situation was considered, namely the operator-valued measure μ was defined on a δ-ring $\mathcal{P}$. Obviously, the whole theory established there is valid also in the σ-algebra case.

The integration theory of vector-valued functions with respect to operator-valued measures was researched by Dobrakov [13] and developed in [15]. Links between Dobrakov's integral theories and other integral theories were presented in [37]. In the situation when $N(S, F) = L(S, F)$, where $L(S, F)$ is the space of all linear bounded operators, then $f \in \mathcal{M}(\mu)$ is equal to the class of integrable functions in the sense of Dobrakov (cf. [13]), and furthermore, both our and Dobrakov integrals are identical. Because every function which is integrable in the sense of Bartle (see [1]) is integrable in the sense of Dobrakov, it also belongs to $\mathcal{M}(\mu)$. In the article [14], Dobrakov refers to the work [48] containing the above concepts and their properties. It should also be noted that in the special case, where $F = S = \mathbb{R}$, if f is Lebesgue integrable, then f is μ-integrable and $\int\limits_E f\, d\mu = \int\limits_E f\, dm$. Conversely, if f is μ-integrable, then f is Lebesgue integrable (cf. [13]). Hildebrandt's now historic article [20] discussing different types of integration may provide a reference point to better situate later results. The problem of integral representation of linear operators on vector-valued function spaces with respect to the corresponding operator-valued measures has been studied by many authors (see [10–12, 33, 34]). The book [12] is devoted in particular to the theory of integration with respect to vector measures of finite semivariation and its applications. In papers [33, 34], σ-smooth operators are studied and relations between these operators and the corresponding operator measure with values in the space $\mathcal{L}(X, Y)$ of all bounded linear operators between Banach spaces. In the paper [32], the author comprehensively presented the principles of Pettis integrability. Paper [47] considers issues of representation of orthogonally additive operators defined on the space of measurable functions by integrals with respect to the nonlinear operator-valued measure. The theorems in Sects. 2 and 3 use Radon-Nikodym-type theorems. Radon-Nikodym theorems in the situation of measures and functions with values in Banach spaces are presented, among others, in [7]. There are also references to earlier papers [28, 30, 46] in the case of Bochner and Pettis integrals and [31] in the case of the Bartle-Dunford-Schwartz integral.

In the current paper, in Sect. 2, we consider the Radon-Nikodym-type theorem on representation of the given nonlinear operator-valued measure μ by the Pettis integral with respect to non-negative finite measure m. In this section, we make an additional assumption that the space F is a weakly complete Banach space. In Sect. 3, we present a theorem on the representation of the operator measure μ by the Bochner integral with respect to a non-negative finite measure m. In this section, we omit the assumption of weak completeness of the space F, but introduce additional assumptions about the measure μ regarding the finiteness of the variation and the local uniform continuity in the variation. Furthermore, we discuss the relation between the integral with respect to μ and the Pettis and Bochner integrals with respect to the scalar measure m. In Sect. 4, we leverage that representation $Tf \equiv \int_X f \, d\mu$ to place it in the framework of nonexpansive-type operators and iterative methods. With some additional assumptions, the operator $\int_X f \, d\mu$ will be nonexpansive and even a strict contraction. Consequently, standard fixed-point algorithms apply directly: Picard in the contractive regime, and Krasnosel'skii-Mann/Halpern iterations for nonexpansive mappings. In this setting, Nemytskii (pointwise) and Urysohn (kernel) operators arise as immediate special cases of this operator.

Remark 1.3 (Standing assumptions) Throughout, (X, Σ, m) is σ−finite and S, F are (real) Banach spaces. We fix $1 < p < \infty$ and write q for its Hölder conjugate. Integrals are with respect to m and we write dm for the measure differential. By a Carathéodory function $\varphi : S \times X \to F$ we mean: measurable in x and (locally uniformly) continuous in r.

2 The Pettis integrals and operator valued measures

Definition 2.1 Let $(X, \mathcal{R}, m)$ be any finite measure space, and F any Banach space. Given a strongly measurable function $g : X \to F$, we say that g is Pettis integrable if for every set $E \in \mathcal{R}$ there exists an element $\mathcal{H}(E)$, such that

$$< z^*, \mathcal{H}(E) >= \int_E < z^*, g(x) > dm,$$

for any $z^* \in F^*$. The element $\mathcal{H}(E)$ is called the Pettis Integral of g in E, and is denoted by $(P) \int_E g \, dm$.

Definition 2.2 A function $g : X \to F$ is *scalarly m-* integrable if $< z^*, g >\in L^1(m)$, for every $z^* \in F^*$.

The following theorem of Vitali's type for the Pettis integrals is taken from the Musiał's paper (Theorem 5.5 p. 552 in [32]). Refer there for further details and proof.

Theorem 2.1 (*Vitali's Type Theorem*). *Assume that F does not contain any isomorphic copy of c_0. If $g : X \to F$ is scalarly m-integrable and there are Pettis integrable functions g_n such that*

$$\lim_n \int_E < z^*, g_n > dm = \int_E < z^*, g > dm,$$

for all $E \in \mathcal{R}$ and each $z^* \in F^*$, then the function g is a strongly measurable Pettis integrable function and

$$\lim_n (P) \int_E g_n \, dm = (P) \int_E g \, dm,$$

weakly for all $E \in \mathcal{R}$.

Theorem 2.2 (*Radon-Nikodym Theorem for the Pettis integrals*). *Let $\nu : \mathcal{R} \to F$ be any σ-additive measure and ν be absolutely continuous with respect to measure m ($\nu \ll m$). The following conditions are equivalent*:

1. *There exists a strongly measurable Pettis integrable function $g_r : \mathcal{R} \to F$ such that*

$$(P) \int_E g_r \, dm = \nu(E) \quad \textit{for all} \quad E \in \mathcal{R}.$$

2. *The vector measure ν has almost locally compact average range.*

Details of such a formulation of the Radon-Nikodym theorem and its proof are provided in [7]. The proof refers to an earlier paper Moedomo and Uhl [30]. We use this theorem of Radon-Nikodym in the proof of the next theorem.

Theorem 2.3 *Let F be a weakly complete Banach space and $\mu : \mathcal{R} \to N(S, F)$ be an operator valued measure such that $\mu \ll m$ and $\mu(\cdot)r$ has almost locally compact average range for every $r \in S$. Then*

1. *there exists a locally uniformly weakly continuous Carathéodory function $\varphi : S \times X \to F$ such that $\varphi(0, x) = 0$ m-almost everywhere and*

$$\mu(E)r = (P) \int_E \varphi(r, x) \, dm$$

   for $r \in S$ and $E \in \mathcal{R}$.

2. *for each $f \in M(\mu, m)$, the function $\varphi(f(\cdot), \cdot)$ is a strongly measurable Pettis integrable function and*

$$\int_E f \, d\mu = (P) \int_E \varphi(f(x), x) \, dm$$

   for every $E \in \mathcal{R}$.

The function φ is unique up to sets of the form $S \times N$ with N an m-null set in X.

Proof It is convenient to divide the proof into several steps.

Step 1. For each $r \in S$, the operator valued measure $\mu : \mathcal{R} \to N(S, F)$ is a vector measure $\mu(\cdot)r : \mathcal{R} \to F$. This vector measure satisfying the assumptions of the Radon-Nikodym Theorem 2.2 and its condition (2) are satisfied. Therefore it follows by this theorem that

$$\mu(E)r = (P)\int_E g_r(x)\, dm,$$

where $r \in S$, $E \in \mathcal{R}$ and $g_r : X \to F$ is a strongly measurable Pettis integrable function.

Step 2. Let $Q \subset S$ be a countable dense subset (since S is separable) and let K_α denote the ball of radius α in S. In this step we verify that $g(r, x) := g_r(x)$ is a uniformly weakly continuous function on the set $Q_\alpha = Q \cap K_\alpha$ for m-almost everywhere $x \in X$ and for each $\alpha > 0$.

Let $z^* \in F^*$ and $\epsilon > 0$. We define

$$\begin{aligned} A_n(\epsilon) =& \{x \in X : \exists r_1, r_2 \in Q_\alpha \text{ such that } \|r_1 - r_2\| \le \frac{1}{n} \\ & \text{and } | < z^*, g(r_1, x) - g(r_2, x) > | \ge \epsilon\}. \end{aligned}$$

First, using general condition (μ1) (see Assumption 1.2), we will prove that $\lim_n m(A_n(\epsilon)) = 0$.

For if $\lim_n m(A_n(\epsilon)) \neq 0$ there exists a positive number a such that $m(A_n(\epsilon)) \ge a$.

Given $\delta > 0$, let n be chosen so that $\frac{1}{n} \le \delta$. Let $\{(p_i, q_i)\}_{i\ge1}$ be an enumeration of all pairs in $Q_\alpha \times Q_\alpha$, such that $\|p_i - q_i\| \le \frac{1}{n}$ and let

$$A_n^i = \{x \in X : | < z^*, g(p_i, x) - g(q_i, x) > | \ge \epsilon\}.$$

Put $A_n^0 = \emptyset$. Similarly as [29] define $C_n^{i+1} = A_n^{i+1} \setminus \bigcup_{j \le i} A_n^j$ for $i = 0, 1, 2, \ldots$. Note that

$$a \le m(A_n(\epsilon)) = \sum_{i=1}^{\infty} m(C_n^i).$$

Let N be so chosen, that $\sum_{i=1}^{N} m(C_n^i) \ge a/2$. We get

$$\sum_{i=1}^{N} \int_{C_n^i} | < z^*, g(p_i, x) - g(q_i, x) > | \, dm \ge \epsilon a/2.$$

Then there exists a sequence of sets (D_n^i) such that $D_n^i \subset C_n^i$ for every $i \in \mathbb{N}$ and

$$\sum_{i=1}^{N} | \int_{D_n^i} < z^*, g(p_i, x) - g(q_i, x) > dm \, | \ge \epsilon a/8.$$

Using the definition of Pettis integral (Definition 2.1)

$$\sum_{i=1}^{N} | < z^*,\ (P)\int_{D_n^i} g(p_i,x)\,dm - (P)\int_{D_n^i} g(q_i,x)\,dm > | \geq \epsilon a/8.$$

$$\sum_{i=1}^{N} | < z^*,\ \mu(D_n^i)p_i - \mu(D_n^i)q_i > | \geq \epsilon a/8.$$

Then there exists a sequence of sets (E_n^i) such that $E_n^i \subset D_n^i$ for every $i \in \mathbb{N}$ and

$$|\sum_{i=1}^{N} < z^*,\ \mu(E_n^i)p_i - \mu(E_n^i)q_i > | \geq \epsilon a/32 > 0.$$

On the other hand we have

$$\| \sum_{i=1}^{N} [\mu(E_n^i)p_i - \mu(E_n^i q_i)] \| \leq s v_\delta(\mu_\alpha, X) \to 0 \quad \text{as} \quad \delta \to 0.$$

Therefore $< z^*, \sum_{i=1}^{N}[\mu(E_n^i)p_i - \mu(E_n^i)q_i] >\to 0$ which contradicts the previous inequality. Consequently, we have that $\lim_n m(A_n(\epsilon)) = 0$.

Thus if $A^\epsilon = \bigcap_{n\geq 1} A_n(\epsilon)$ then $m(A^\epsilon) = 0$. Now let $B_\alpha = \bigcup_{j\geq 1} A^{1/j}$, obviously $m(B_\alpha) = 0$ and for every $\alpha > 0$ function $g(\cdot,x)$ is uniformly weakly continuous on the set Q_α for every $x \in X \setminus B_\alpha$.

Step 3. Let's take ball K_n for $\alpha = n$. For each such a ball K_n we have a corresponding set $B_n \subset X$.

For $r \in Q$ we define

$$\varphi(r,x) = \begin{cases} g(r,x) & \text{for } x \in X \setminus \bigcup_{n\geq 1} B_n \\ 0 & \text{for } x \in \bigcup_{n\geq 1} B_n \end{cases}$$

We extend $\varphi(r,x)$ to $r \in S$. Let $r \in S$ and $r_n \to r$, $r_n \in Q$ for every $n \in \mathbb{N}$ (by density of Q in S). Let $\alpha > 0$ and $r \in K_\alpha$. Because $g(\cdot,x)$ is uniformly weakly continuous on Q_α, for each $x \in X \setminus \bigcup_{n\geq 1} B_n$, it follows that $\{\varphi(r_n,x)\}$ is weak-Cauchy for m-almost all x. Since F is weakly complete, it follows that there exists $w - \lim_n \varphi(r_n,x)$. We define $\varphi(r,x) := w - \lim_n \varphi(r_n,x)$.

Let us first note that the function $\varphi(\cdot,x)$ is locally weakly uniformly continuous for every $x \in X \setminus B_\alpha$. Let $\alpha > 0$, $\varepsilon > 0$ and $z^* \in F^*$. We will show that there is a $\delta > 0$ such that

$$| < z^*, \varphi(r,x) - \varphi(s,x) > | < \epsilon$$

for $r,s \in K_\alpha$ such that $\|r - s\| < \delta$.

Indeed, choose $\delta_1 > 0$ such that

$$| < z^*, \varphi(p,x) - \varphi(q,x) > | < \frac{\epsilon}{3}$$

for $p,q \in Q_\alpha$ and $\|p-q\| < \delta_1$. Fix $r,s \in K_\alpha$. From the density of the set Q_α in K_α, we can select r_α, $s_\alpha \in Q_\alpha$ such that $\|r_\alpha - r\| < \delta_1$, $\|s_\alpha - s\| < \delta_1$ and

$$|<z^*, \varphi(r_\alpha, x) - \varphi(r,x)>| < \frac{\epsilon}{3}$$
$$|<z^*, \varphi(s_\alpha, x) - \varphi(s,x)>| < \frac{\epsilon}{3}.$$

Because for every $\alpha > 0$ function $g(\cdot, x)$ is uniformly weakly continuous on the set Q_α for every $x \in X \setminus B_\alpha$ and r_α, $s_\alpha \in Q_\alpha$ hence

$$|<z^*, \varphi(r_\alpha, x) - \varphi(s_\alpha, x)>| < \frac{\epsilon}{3},$$

for $\|r_\alpha - s_\alpha\| < \delta_2$. Choose $\delta > 0$ and $\delta \leq \min(\delta_1, \delta_2)$, hence we have $\|r - s\| < \delta$, $\|r_\alpha - s_\alpha\| < \delta$ and all the above inequalities are satisfied. Hence

$$\begin{aligned} |<z^*, \varphi(r,x) - \varphi(s,x)>| \ &\leq |<z^*, \varphi(r_\alpha, x) - \varphi(r,x)>| + \\ &|<z^*, \varphi(s_\alpha, x) - \varphi(s,x)>| + \\ &|<z^*, \varphi(r_\alpha, x) - \varphi(s_\alpha, x)>| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

We will further show that the functions $\varphi(r,x) = g(r,x)$ m-almost everywhere.

Because the operator measure μ is locally uniformly continuous in semivariation on every $E \in \mathcal{R}$ then for every $r_n, r_m \in K_\alpha$ such that $\|r_n - r_m\| \leq \delta$

$$\|\mu(E)r_n - \mu(E)r_m\| \leq sv_\delta(\mu_\alpha, E) \to 0 \quad \mathit{if} \quad \delta \to 0.$$

Therefore for $\|r_n - r\|_S \to 0$, $\|\mu(E)r_n - \mu(E)r\| \to 0$ uniformly on K_α.

In particular $\mu(E)r_n \to \mu(E)r$ in weak topology, that is $<z^*, \mu(E)r_n> \to <z^*, \mu(E)r>$ for any $z^* \in F^*$.

On the other hand, from Radon-Nikodym Theorem and by definition of Pettis integrals, we have

$$<z^*, \mu(E)r_n> = <z^*, (P)\int_E g(r_n, x)\, dm> = \int_E <z^*, g(r_n, x)> dm,$$

and

$$<z^*, \mu(E)r> = <z^*, (P)\int_E g(r,x)\, dm> = \int_E <z^*, g(r,x)> dm.$$

That is

$$\int_E <z^*, g(r_n, x)> dm \to \int_E <z^*, g(r,x)> dm.$$

But for every $E \in \mathcal{R}$

$$\lim_n \int_E < z^*, g(r_n, x) > dm = \lim_n \int_E < z^*, \varphi(r_n, x) > dm$$

since $r_n \in \mathbb{Q}$. So

$$\int_E < z^*, g(r, x) > dm = \lim_n \int_E < z^*, g(r_n, x) > dm = \lim_n \int_E < z^*, \varphi(r_n, x) > dm.$$

Hence

$$\lim_n \int_E < z^*, \varphi(r_n, x) > dm = \int_E < z^*, g(r, x) > dm$$

for every $E \in \mathcal{R}$.

Because m is a finite measure, hence there exists (r_{n_k}) subsequence of (r_n) such that

$$< z^*, \varphi(r_{n_k}, x) > \to < z^*, g(r, x) >$$

m-almost everywhere. On the other hand, as shown above,

$$< z^*, \varphi(r_n, x) > \to < z^*, \varphi(r, x) >$$

m-almost everywhere. By the uniqueness of the weak-limit we finally get

$$\varphi(r, x) = g(r, x)$$

for m-almost everywhere x and

$$\lim_n \int_E < z^*, \varphi(r_n, x) > dm = \int_E < z^*, \varphi(r, x) > dm$$

for every $E \in \mathcal{R}$.

Step 4. Using the facts that $\varphi(0, x) = 0$ m-almost everywhere and that $\mu(E)r = \int_E \varphi(r, x)\, dm$ for every $E \in \mathcal{R}$, we can show immediately that

$$\int_E s\, d\mu = (P) \int_E \varphi(s(x), x)\, dm$$

for every $\mathcal{R} - \textit{simple}$ function s.

Step 5. Let $f \in M(\mu, m)$ and (s_n) be a sequence of $\mathcal{R} - \textit{simple}$ functions converging to f m-almost everywhere, such that $\lim_n \int_E s_n\, d\mu$ exists for each $E \in \mathcal{R}$ and $\lim_n \int_E s_n\, d\mu = \int_E f\, d\mu$. Since a sequence converging at F is also weakly convergent at F, then

$$\int_E s_n\, d\mu \to \int_E f\, d\mu$$

weakly for every $E \in \mathcal{R}$. Hence

$$< z*, \int_E s_n \, d\mu > \quad \to \quad < z^*, \int_E f \, d\mu >$$

On the other hand, from Step 4, we have

$$\int_E s_n \, d\mu \;= (P) \int_E \varphi(s_n(x), x) \, dm$$

for each $n \in \mathbb{N}$ and every $E \in \mathcal{R}$.

From the definition of the Pettis integral we have

$$< z^*, \int_E s_n \, d\mu > = < z^*, (P) \int_E \varphi(s_n(x), x) \, dm > = \int_E < z^*, \varphi(s_n(x).x) > dm.$$

So

$$\int_E < z^*, \varphi(s_n(x).x) > dm \quad \to \quad < z^*, \int_E f \, d\mu > .$$

On the other hand from Vitali's theorem in $L^1(m)$ (see for example [4] th. 4.5.6 p. 269) follows that if $< z^*, \varphi(s_n(x), x) > \in L^1(m)$ for every $n \in \mathbb{N}$ and $\lim_n \int_E < z^*, \varphi(s_n(x), x) > \; dm$ exists for every $E \in \mathcal{R}$, then, it follows in particular, the exist function $\lambda \in L^1(m)$ such that $\lim_n \int_E < z^*, \varphi(s_n(x), x) > dm = \int_E \lambda(x) \, dm$ for every $E \in \mathcal{R}$. However $m(X) < \infty$, then there exists subsequence $(s_{n_k}(x))$ such that

$$< z^*, \varphi(s_{n_k}(x), x) > \to \lambda(x)$$

m-almost everywhere. But

$$< z^*, \varphi(s_n(x), x) > \to < z^*, \varphi(f(x), x) >$$

m-almost everywhere. Hence, by the uniqueness of weak-limit, finally we get

$$< z^*, \varphi(f(x), x) > = \lambda(x)$$

m-almost everywhere and therefore $< z^*, \varphi(f(x), x) > \in L^1(m)$ and

$$\int_E < z^*, \varphi(s_n(x).x) > dm \quad \to \quad \int_E < z^*, \varphi(f(x).x) > dm$$

for every $E \in \mathcal{R}$ and every $z^* \in F^*$.

Note that the assumptions of Theorem 2.1 are satisfied, because

1. from the fact that F is weakly complete, it follows that it does not contain a subspace isomorphic with c_0 (Bessaga-Pełczyński [3]),

2. from the fact that then $\int_E < z^*, \varphi(f(x),x) > dm \in L^1(m)$ for every $z^* \in F^*$ and

$$\int_E < z^*, \varphi(s_n(x),x) > dm \to \int_E < z^*, \varphi(f(x),x) > dm,$$

for every $z^* \in F^*$, for $m - almost$ everywhere $x \in X$ and each $E \in \mathcal{R}$.

From Theorem 2.1 we know that under such assumptions if

$$\lim_n \int_E < z^*, \varphi(s_n(x),x) > dm = \int_E < z^*, \varphi(f(x),x) > dm,$$

for all $E \in \mathcal{R}$ and each $z^* \in F^*$, then function $\varphi(f(\cdot),\cdot)$ is a strongly measurable Pettis integrable function and

$$\lim_n (P) \int_E \varphi(s_n(x),x)\, dm = (P) \int_E \varphi(f(x),x)\, dm,$$

weakly for all $E \in \mathcal{R}$. Thus, ultimately

$$\int_E f\, d\mu = (P) \int_E \varphi(f(x),x)\, dm$$

for every $E \in \mathcal{R}$.

The function φ is unique up to sets of the form $S \times N$ with N is m-null set in X.

The proof of uniqueness.

Assume that there are two locally uniformly weakly continuous Carathéodory functions $\varphi_1, \varphi_2 : S \times X \to F$ such that for a fixed $r \in S$, the following holds for all $E \in \mathcal{R}$:

$$\mu(E)r = (P) \int_E \varphi_1(r,x)\, dm = (P) \int_E \varphi_2(r,x)\, dm.$$

By the definition of the Pettis integral, for every $z^* \in F^*$, this implies the equality of the corresponding scalar integrals:

$$\int_E \langle z^*, \varphi_1(r,x) \rangle\, dm = \int_E \langle z^*, \varphi_2(r,x) \rangle\, dm.$$

This means that for every $z^* \in F^*$,

$$\int_E \langle z^*, \varphi_1(r,x) - \varphi_2(r,x) \rangle\, dm = 0$$

for all $E \in \mathcal{R}$. By uniqueness of signed measures (via the monotone class theorem), if $\int_E g\, dm = 0$ for all $E \in \mathcal{R}$, then $g = 0$ m-a.e. Thus, for each $z^* \in F^*$, we have $\langle z^*, \varphi_1(r,x) - \varphi_2(r,x) \rangle = 0$ for m-almost every $x \in X$. By the Hahn-Banach theorem, if a vector v satisfies $\langle z^*, v \rangle = 0$ for all $z^* \in F^*$, then v must be the zero vector. Applying this pointwise for a.e. $x \in X$, we conclude that $\varphi_1(r,x) = \varphi_2(r,x)$ for m-almost every $x \in X$.

In the case of the Pettis integral, a weaker form of uniqueness is typically proven, known as "Pettis almost everywhere" (Pettis a.e.) uniqueness. This means there exists a set N

of measure zero ($\mu(N) = 0$) such that for all points x outside this set ($x \notin N$), the kernels $\varphi_1(x,r)$ and $\varphi_2(x,r)$ are equal, but in the sense of the weak topology. In practice, this amounts to the condition:

$$\forall x \notin N, \forall r \in \mathbb{R}, \forall g^* \in F^* : \langle g^*, \varphi_1(x,r)\rangle = \langle g^*, \varphi_2(x,r)\rangle$$

This weaker condition is a natural consequence of the proof in general Banach spaces. To obtain stronger uniqueness (in the norm sense), additional assumptions are needed, such as the separability of the space F. □

Remark 2.1 (Separable $F \Rightarrow$ a common null set) If the Banach space F is separable, then there exists a single m-null set $N \subset X$ such that $\varphi_1(r,x) = \varphi_2(r,x)$ for all $r \in S$ and all $x \in X \setminus N$. Indeed, separability of F yields a countable total set $\{z_n^*\}_{n\in\mathbb{N}} \subset F^*$. Fix also a countable dense set $D \subset S$ (e.g. when $S \subset \mathbb{R}^k$). For each n and $r \in D$, apply the above Pettis-uniqueness argument to get a null set $N_{n,r}$ on which $\langle z_n^*, \varphi_1(r,\cdot) - \varphi_2(r,\cdot)\rangle = 0$. Let $N := \bigcup_n \bigcup_{r\in D} N_{n,r}$; then $m(N) = 0$. For $x \notin N$ and every n, the map $r \mapsto \langle z_n^*, \varphi_1(r,x) - \varphi_2(r,x)\rangle$ is continuous (local uniform weak continuity of Carathéodory functions) and vanishes on the dense set D, hence it vanishes on all S. Since $\{z_n^*\}$ is total, we conclude $\varphi_1(r,x) = \varphi_2(r,x)$ for all $r \in S$ and all $x \in X \setminus N$.

Corollary 2.1 *If additionally one assumes $m \ll \mu$ then*

$$\int_E f\, d\mu = \int_E \varphi(f(x),x)\, dm$$

holds for every $f \in \mathcal{M}(\mu)$ and all $E \in \mathcal{R}$.

3 The Bochner integrals and operator valued measures

Definition 3.1 Let $(X, \mathcal{R}, m)$ be a finite measure space, and F a Banach space.

1. A function $s : X \to F$ it said to be *simple* if it is of the form $s = \sum_{i=1}^N x_i \cdot 1_{A\,i}$ where $x_i \in F$ and $A_i \in \mathcal{R}$ for all $i = 1, 2, \dots, N$.
2. The integral of a simple function s is the element: $\int s\, dm = \sum_{i=1}^{N} x_i \cdot m(A_i) \in F$.
3. We say that (g_n) converges *in measure* to some function g, if the following holds:

   $$\lim_{n\to\infty} m(\{x \in X : \|g_n(x) - g(x)\| > \epsilon\}) = 0.$$

4. In case $g : X \to F$ is the limit in measure of some sequence (s_n) of simple functions, then g is said to be m-measurable (also, strongly m-measurable).
5. We say that g is a Bochner integrable, if there exists a sequence of simple functions (s_n), converging in measure to g, and such that

   $$\lim_{(n,k)\to\infty} \int \|s_n - s_k\|\, dm = 0.$$

   If $g : X \to F$ is Bochner integrable, then the sequence $(\int s_n\, dm)$ convergent in F, and the limit is independent of the particular sequence (s_n).
6. The limit $\int s_n\, dm$ called the Bochner integral of g and is denoted by $\int g\, dm$.

Theorem 3.1 (The Radon-Nikodym Theorem for the Bochner integrals) *Let $(X, \mathcal{R}, m)$ be a σ-finite positive measure space and let F be a Banach space. Let ϱ be an F-valued measure on X. Then ϱ is the indefinite integral with respect to m of a F-valued Bochner integrable function on X if and only if*

1. *ϱ is m-continuous, that is $\varrho(E) = 0$ whenever $m(E) = 0$, $E \in \mathcal{R}$,*
2. *ϱ has almost locally compact average range,*
3. *the total variation of ϱ is finite, $V(\varrho)(X) < +\infty$.*

This standard theorem for Bochner integrals is formulated and proved in the papers in [28] [th. 8] and [46]). We refer there for more details.

Let us now formulate the lemma used in the proof of the following theorem.

Lemma 3.1 *Let $g : S \to \mathcal{L}^1_F$ where $\mathcal{L}^1_F$ is the space of all F-valued Bochner integrable functions on X, S is separable Banach space. For $r \in S$ denote the indefinite integral of $g(r)$ by $\int g(r)\,dm$. Then if the family $\{\int g(r)\,dm\}_{r \in S}$ is locally uniformly continuous in variation there exists a Carathéodory function $\varphi : S \times X \to F$ such that, $g(r)(x) = \varphi(r, x)$ m-almost everywhere.*

Proof The proof is analogous to the proof of Lemma 3 in paper [29]. Consider that the proof in [29] was for the case $S = \mathbb{R}$. However, only the separability of the real line was used in this the proof. We assume separability of the space S, so the whole construction of the proof does not change. Steps 2 and 3 in the proof of the theorem 2.3 are also based on the same methods. □

Theorem 3.2 *Let $\mu : \mathcal{R} \to N(S, F)$ be an operator valued measure satisfying the following conditions:*

1. *$\mu \ll m$,*
2. *$\mu(\cdot)r$ has almost locally compact average range for every $r \in S$,*
3. *$V(\mu(\cdot)r)(X) < +\infty$ for every $r \in S$,*
4. *μ is locally uniformly continuous in variation.*

Then there exists a locally uniformly Carathéodory function $\varphi : S \times X \to F$ such that $\varphi(0, x) = 0$ m-almost everywhere and for every $r \in S$, $E \in \mathcal{R}$

$$\mu(E)r = \int_E \varphi(r, x)\,dm.$$

In addition if f is a totally $\mathcal{R}$-measurable function, then the function $\varphi(f(\cdot), \cdot)$ is Bochner integrable and

$$\int_E f\,d\mu = \int_E \varphi(f(x), x)\,dm$$

for every $E \in \mathcal{R}$. The function φ is unique up to sets of the form $S \times N$ with N is a m-null set in X.

Proof Since the operator valued measure μ satisfies condition (1) - (3) we can use the Radon-Nikodym Theorem 3.1. It follows that for every $r \in S$ and $E \in \mathcal{R}$

$$\mu(E)r = \int_E g_r(x)\, dm,$$

where g_r is a Bochner integrable function. From the Lemma 3.1, using the condition (4) we may prove that exists locally uniformly continuous Carathéodory function $\varphi : S \times X \to F$ such that, for every $r \in S$, $\varphi(r,x) = g_r(x)$ m-almost everywhere and

$$\mu(E)r = \int_E \varphi(r,x)\, dm$$

for $r \in S$, $E \in \mathcal{R}$. Using the fact that $\varphi(0,x) = 0$ m-almost everywhere and that $\mu(E)r = \int_E \varphi(r,x)\, dm$ for every $E \in \mathcal{R}$, we can show immediately that

$$\int_E s\, d\mu = \int_E \varphi(s(x),x)\, dm$$

for every $\mathcal{R}$-simple function s and every $E \in \mathcal{R}$. Let f be *totally* $\mathcal{R}$-measurable function. It follows then there exists a sequence of $\mathcal{R}$-simple functions (s_n) uniformly converging to f. For fixed $s_n = \sum_{i=1}^{N} r_i \cdot 1_{B_i}$ and $s_m = \sum_{j=1}^{M} r'_j \cdot 1_{Cj}$ we define a set function G by

$$G(E) = \int_E s_n\, d\mu - \int_E s_m\, d\mu.$$

For convenience of reference we note that if $g : X \to F$ is a function in L^1_F and if, for each $E \in \mathcal{R}$, $G(E) = \int_E g\, dm$, then $V(G)(E) = \int_E \|g(x)\|\, dm(x)$. For a proof of this fact we refer to Dunford and Schwartz [16]. From this we obtain

$$V(G)(E) = \int_E \|\varphi(s_n(x),x) - \varphi(s_m(x),x)\|\, dm,$$

because

$$G(E) = \int_E [\varphi(s_n(x),x) - \varphi(s_m(x),x)]\, dm.$$

On the other hand we have

$$\begin{aligned} V(G)(E) &= \sup_{(E_k)}\{\sum_{k=1}^{K} \| \int_{E_k} s_n\, d\mu - \int_{E_k} s_m\, d\mu \|\} \\ &= \sup_{(E_k)}\{\sum_{k=1}^{K} \| \sum_{i=1}^{N} \mu(B_i^k) r_i - \sum_{j=1}^{M} \mu(C_j^k) r'_j \|\}, \end{aligned}$$

where $B_i^k = E_k \cap B_i$ and $C_j^k = E_k \cap C_j$.

By taking all intersections of the form $B_i^k \cap C_j^k$ for $i = 1, 2, \ldots, N$, $j = 1, \ldots, M$ and reindexing, we get sets (D_p^k) $p = 1, 2, \ldots, N \cdot M$. Put $r_p = s_n(D_p^k)$ and $r'_p = s_m(D_p^k)$. We obtain

$$\begin{aligned} V(G)(E) &= \sup_{(E_k)}\{\sum_{k=1}^{K} \| \sum_{p=1}^{N\cdot M}(\mu(D_p^k)r_p - \mu(D_p^k)r'_p)\|\} \\ &\le \sup\{\sum_{k=1}^{K} V(\mu(\cdot)r_k - \mu(\cdot)r'_k)(E_k)\} \\ &= v_\delta(\mu_\alpha, E). \end{aligned}$$

The supremum is taken over all finite pairwise disjoint measurable partitions of $E \in \mathcal{R}$ and over all pairs r_k, r'_k such that $\|r_k\| \le \alpha$, $\|r'_k\| \le \alpha$, $\|r_k - r'_k\| \le \delta$, where $\delta = \sup_{n,m} \|s_n(x) - s_m(x)\|$ and $\alpha > 0$ such that $s_n(x) \in K_\alpha$, $s_m(x) \in K_\alpha$ for every $x \in X$. Existence of such α $(0 < \alpha < +\infty)$ is caused by the fact that each $\mathcal{R}$-simple function s_n is bounded and sequence of functions (s_n) uniformly converging to f. By the same fact it follows that $\delta \to 0$ for $n, m \to +\infty$. Because μ is locally uniformly continuous in variation, we get $V_\delta(\mu_\alpha, E) \to 0$.

Hence

$$\int_E \|\varphi(s_n(x), x) - \varphi(s_m(x), x)\| \, dm \to 0$$

and consequently

$$\int_E \varphi(s_n(x), x) \, dm \to \int_E \varphi(f(x), x) \, dm.$$

From the definition of integral of *totally* $\mathcal{R}$-measurable function [cf. [48]] we obtain

$$\int_E s_n \, d\mu \to \int_E f \, d\mu.$$

Finally we have

$$\int_E f \, d\mu = \int_E \varphi(f(x), x) \, dm,$$

for every $E \in \mathcal{R}$ and each *totally* $\mathcal{R}$-measurable f.

The proof of uniqueness.

Assume that there are two locally uniformly continuous Carathéodory functions $\varphi_1, \varphi_2 : S \times X \to F$ such that for a fixed $r \in S$, the following holds for all $E \in \mathcal{R}$:

$$\mu(E)r = \int_E \varphi_1(r, x) \, dm = \int_E \varphi_2(r, x) \, dm.$$

This implies that the integral of their difference is zero for all measurable sets:

$$\int_E [\varphi_1(r, x) - \varphi_2(r, x)] \, dm = 0 \qquad (\forall E \in \mathcal{R}).$$

A key property of the Bochner integral, based on convergence in norm, is that if a Bochner integrable function has a zero integral over all measurable sets, then the function itself must be zero almost everywhere. Applying this principle directly to the difference, we conclude that $\varphi_1(r,x) - \varphi_2(r,x) = 0$ for m-almost every $x \in X$. Therefore, $\varphi_1 = \varphi_2$ m-almost everywhere. □

Remark 3.1 (Bochner case: a common null set without separability) In the Bochner setting one does not need separability of F to obtain a single m-null set $N \subset X$ such that $\varphi_1(r,x) = \varphi_2(r,x)$ for all $r \in S$ and all $x \in X \setminus N$. Indeed, for each r fix a null set N_r with $\varphi_1(r,\cdot) = \varphi_2(r,\cdot)$ m-a.e. on $X \setminus N_r$ (equality in F-norm). Choose a countable dense set $D \subset S$ and put $N := \bigcup_{r\in D} N_r$, so $m(N) = 0$. For every $x \notin N$ we have $\varphi_1(r,x) = \varphi_2(r,x)$ for all $r \in D$. If, as assumed, the Carathéodory regularity holds in r (for a.e. x the map $r \mapsto \varphi_i(r,x)$ is continuous—norm or just weak continuity suffices), then by density of D this equality extends to all $r \in S$. Hence a common null set exists irrespective of separability of F.

Theorem 3.3 *Conversely, for every Bochner integrable locally uniformly continuous Carathéodory function $\varphi : S \times X \to F$ such that $\varphi(0,x) = 0$ m-almost everywhere, a set function μ defined by the formula $\mu(E)r = \int_E \varphi(r,x)\,dm$ is an operator valued measure satisfying the conditions* (1)-(4) *from Theorem* 3.2.

Proof Let μ be defined by the formula $\mu(E)r = \int_E \varphi(r,x)\,dm$ and $\varphi : S \times X \to F$ be a locally uniformly continuous Carathéodory function such that $\varphi(0,x) = 0$ m-almost everywhere. Obviously, such measure satisfies conditions (1)-(3) by the Radon-Nikodym theorem 3.1. Therefore, it is sufficient to prove (4). Assume to the contrary that $v_\delta(\mu_\alpha.X) \nrightarrow 0$ as $\delta \to 0$. Since

$$v_\delta(\mu_\alpha.X) = \sup\{\sum_{i=1}^{n} \int_{A_i} \|\varphi(r_i,x) - \varphi(r_i',x)\|\, dm\}$$

we may assume that there exists an $a > 0$, disjoint sets (A_i) such that $\cup_{i=1}^{n} A_i = X$ such that $r_i, r_i' \in K_\alpha$, $\|r_i - r_i'\| \leq 0$ and

$$\sum_{i=1}^{n} \int_{A_i} \|\varphi(r_i,x) - \varphi(r_i',x)\|\, dm \; > a.$$

On the other hand φ is uniformly continuous on every ball K_α for m-almost every $x \in X$. Since $r_i, r_i' \in K_\alpha$, it follows that we can choose such $\delta > 0$ that for $\|r_i - r_i'\| \leq \delta$ we have $\|\varphi(r_i,x) - \varphi(r_i',x)\| \leq \frac{a}{n \cdot m(X)}$. Hence,

$$\sum_{i=1}^{n} \int_{A_i} \|\varphi(r_1,x) - \varphi(r_1',x)\|\, dm \leq a.$$

Contradiction. □

Theorem 3.4 *If μ is an operator measure satisfying conditions* (1)-(4) *from Theorem* 3.2 *and φ is a function existing by this theorem, then*

1. *for every* $f \in M(\mu, m)$ $\varphi(f(\cdot), \cdot)$ *is Pettis integrable and* $\int_E f\,d\mu = (P)\int_E \varphi(f(x), x)\,dm$ (*Pettis integral*),
2. *for every* $f \in MV(\mu, m)$ $\varphi(f(\cdot), \cdot)$ *is Bochner integrable and* $\int_E f\,d\mu = \int_E \varphi(f(x), x)\,dm$ (*Bochner integral*),

for every $E \in \mathcal{R}$.

Proof Since $f \in M(\mu, m)$ then

$$\int_E f\,d\mu = \lim_k \int_E f \cdot 1_{Z_k}\,d\mu,$$

where (Z_k) is non-decreasing sequence of sets such that $(X \setminus Z_k) \searrow N$, N is m-null, $f \cdot 1_{Z_k}$ is *totally* $\mathcal{R}$-measurable function and $\lim_k \int_E f \cdot 1_{Z_k}\,d\mu$ exists for every $E \in \mathcal{R}$. Existence of such sets (Z_k) follows from Remark 1.1. By Theorem 3.2 we have

$$\int_E f \cdot 1_{Z_k}\,d\mu = \int_E \varphi(f \cdot 1_{Z_k}(x), x)\,dm.$$

Since the limit $\lim_k \int_E \varphi(f \cdot 1_{Z_k}(x), x)\,dm$ exists for every $E \in \mathcal{R}$ and $\varphi(f \cdot 1_{Z_k}(x), x) \to \varphi(f(x), x)$ m-almost everywhere, then

$$\int_E f\,d\mu = \int_E \varphi(f(x), x)\,dm,$$

where this integral is the Pettis integral (the second Dunford integral) cf. [16].

If $f \in MV(\mu, m)$ then $V(G_f)(X) < +\infty$ and

$$\int_E \|\varphi(f \cdot 1_{Z_k}(x), x)\|\,dm = V(G_f)(Z_k) \le V(G_f)(X) < +\infty.$$

Thus, $\{\|\varphi(f \cdot 1_{Z_k}(x), x)\|\}$ is a sequence of Bochner integrable functions. On the other side, this sequence is non-decreasing. Therefore, applying Lebesgue monotone convergence theorem, we obtain

$$\int_E \|\varphi(f(x), x)\|\,dm = \lim_k \int_E \|\varphi(f \cdot 1_{Z_k}(x), x)\|\,dm \le V(G_f)(X) < +\infty.$$

Finally, $\varphi(f(\cdot), \cdot)$ is the Bochner integrable and $\int_E f\,d\mu = \int_E \varphi(f(x), x)\,dm$. □

Remark 3.2 Part (a) of Theorem 3.4 corresponds to the Theorem 2.3, in which assumptions (3) and (4) were omitted, but additionally F was assumed to be weakly complete Banach space. As a result of that theorem we got a kernel function φ, which was only locally uniformly weakly Carathéodory function.

4 Application to nonexpansive-type operators and iterative methods

By Sects. 2–3, the Pettis/Bochner Radon-Nikodym representation of μ provides a Carathéodory kernel φ so that, for every $E \in \mathcal{R}$ and for every totally $\mathcal{R}$-measurable $f : X \to S$ we have the measure representation

$$\int_E f \,\mathrm{d}\mu = \int_E \varphi\big(f(x), x\big)\,\mathrm{d}m(x).$$

Definition 4.1 We define the *global* nonlinear integral operator as

$$Tf := \int_X f\, d\mu$$

for every totally $\mathcal{R}$-measurable $f : X \to S$.

As a direct consequence of Theorem 3.2 (with the same notation and the same assumptions))

$$Tf = \int_X \varphi\big(f(y), y\big)\, dm(y) \in F$$

which is well-defined whenever $y \mapsto \varphi(f(y), y)$ is Bochner integrable.

4.1 Nonexpansive/contractive regime from Theorem 3.2

Assumption 4.1 (Lipschitz modulus) There exists a measurable function $L : X \to [0, \infty)$ such that for m-a.e. $y \in X$ and all $r_1, r_2 \in S$,

$$\|\varphi(r_1, y) - \varphi(r_2, y)\| \le L(y)\, \|r_1 - r_2\|.$$

Moreover, for $p \in [1, \infty]$ with conjugate q, let $L \in L^q(X)$.

We assume the following Lipschitz control, stated with the same kernel φ as in Theorem 3.2:

(L1) For m-a.e. $y \in X$, the map $r \mapsto \varphi(r, y)$ is Lipschitz on every ball $K_\alpha \subset S$, with constant $L(y)$; that is,

$$\|\varphi(r_1, y) - \varphi(r_2, y)\| \le L(y)\, \|r_1 - r_2\| \qquad (\forall\, r_1, r_2 \in S). \tag{1}$$

(L2) Let $p \in [1, \infty]$ and q be its conjugate exponent ($1/p + 1/q = 1$; by convention $q = \infty$ if $p = 1$). Assume $L \in L^q(X)$ and $\|L\|_{L^q(X)} \le 1$.

(L3) $\varphi(0, y) = 0$ for m-a.e. $y \in X$.

Then for all $f, g \in L^p(X, S)$ (using the Hölder inequality),

$$\|Tf - Tg\| \le \int_X \|\varphi(f(y), y) - \varphi(g(y), y)\|_F \, dm(y) \le \|L\|_{L^q(X)}\, \|f - g\|_{L^p(X,S)}. \tag{2}$$

Consequently, T is nonexpansive if $\|L\|_{L^q(X)} \le 1$, and a strict contraction if $\|L\|_{L^q(X)} < 1$.

This shows that the integral operators arising from Theorem 3.2, when endowed with natural Lipschitz control, belong to the class of nonexpansive-type mappings widely studied in fixed point theory and iterative methods [6, 17, 18, 38–40, 43, 44]

4.2 Iterative schemes for nonexpansive mappings

Assume $F = S$ and $\mathrm{Fix}(T) \neq \emptyset$, where $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

1. Krasnoselskii–Mann iteration. Let $(\alpha_n) \subset (0,1)$ and define

   $$f_{n+1} = (1-\alpha_n)f_n + \alpha_n Tf_n, \qquad n \geq 0.$$

   This scheme goes back to [25, 27]. In uniformly convex spaces, if $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and T is nonexpansive, then (f_n) is bounded and every weak cluster point lies in $\mathrm{Fix}(T)$; see [17, 27]. Under additional assumptions (Opial property or demiclosedness of $I - T$ at 0) one obtains weak convergence of the whole sequence; cf. [5, 36, 39, 45].
2. Halpern iteration (Hilbert case). Let S be a Hilbert space, fix $u \in S$, and let (λ_n) satisfy $\lambda_n \downarrow 0$ and $\sum_n \lambda_n = \infty$. Define

   $$f_{n+1} = \lambda_n u + (1-\lambda_n) Tf_n.$$

   Then $f_n \to P_{\mathrm{Fix}(T)}u$ in norm [2, 19, 39, 41, 42, 44], where $P_{\mathrm{Fix}(T)}$ denotes the metric projection onto $\mathrm{Fix}(T)$.
3. Picard iteration. If T is a strict contraction (i.e. $\|L\|_{L^q(X)} < 1$), then the Picard iteration

   $$f_{n+1} = Tf_n$$

   converges in norm to the unique fixed point of T [6].

4.3 Continuity and convergence

We use the Radon-Nikodym representation from Sects. 2–3,

$$\int_E f\,d\mu = \int_E \varphi(f(x),x)\,dm(x) \qquad (E \in \mathcal{R}), \tag{3}$$

with a Carathéodory kernel φ (Pettis/Bochner RN: Thm. 2.3/3.2/3.4), and the Lipschitz estimate of Assumption 4.1,

$$\|Tf - Tg\| \;\leq\; \|L\|_{L^q}\,\|f-g\|_{L^p}. \tag{4}$$

These two facts are the sole inputs below.

Proposition 4.1 (Norm convergence via the Bochner RN representation) *Assume* 4.1 (*Lipschitz on balls with modulus* $L \in L^q$, $1/p + 1/q = 1$, *and* $\varphi(0,\cdot) = 0$). *If* $f_n \to f_\star$ *in* $L^p(X;S)$, *then*

$$Tf_n \to Tf_\star \quad \textit{in } F, \qquad \textit{and} \qquad \int_E f_n\,d\mu \to \int_E f_\star\,d\mu \textit{ in } F \textit{ for every } E \in \mathcal{R}.$$

Proof From (4), $\|Tf_n - Tf_\star\| \leq \|L\|_{L^q}\|f_n - f_\star\|_{L^p} \to 0$. Using (3) and Hölder,

$$\left\| \int_E \varphi(f_n,x)\,dm - \int_E \varphi(f_\star,x)\,dm \right\| \leq \|L\|_{L^q}\,\|f_n - f_\star\|_{L^p(E;S)} \to 0. \qquad \square$$

Proposition 4.2 (Weak (Pettis) compatibility via the Pettis RN representation) *Under the weak hypotheses of Sect.* 2 (*local uniform weak continuity in r as in Thm.* 2.3) *together with the same Lipschitz control on balls, if* (f_n) *is bounded in* $L^p(X;S)$ *and* $f_n \rightharpoonup f_\star$ *in* L^p, *then for every* $E \in \mathcal{R}$ *and every* $z^* \in F^*$,

$$\left\langle z^*, \int_E f_n \, d\mu \right\rangle \longrightarrow \left\langle z^*, \int_E f_\star \, d\mu \right\rangle.$$

Equivalently, the partial integrals converge weakly in the Pettis sense.

Remark 4.1 Whenever an algorithm in Sect. 4.2 (Picard in the strictly contractive regime; Krasnosel'skiĭ–Mann/Halpern under their standard assumptions) delivers $f_n \to f_\star$ in L^p, Proposition 4.1 yields $Tf_n \to Tf_\star$ and, for every E, $\int_E f_n \, d\mu \to \int_E f_\star \, d\mu$. In the self-mapping case $F = S$, if the limit is a fixed point ($f_\star \in \mathrm{Fix}(T)$), then $Tf_\star = f_\star$. If only weak convergence is available, Proposition 4.2 ensures Pettis-type compatibility for all partial integrals.

Remark 4.2 The abstract integral with respect to a nonlinear operator-valued measure becomes algorithmically tractable: (i) the nonexpansive estimate (4) gives stability of outputs with respect to inputs; (ii) the RN representation (3) guarantees that limits produced by iterative schemes are consistent with the measure-level objects $\int_E (\cdot) \, d\mu$. This closes the loop between Sects. 2–3 (representation) and Sect. 4.2 (iterations) without any further structure.

4.4 Examples: Nemytskii and Urysohn-type operators

All integral interchanges below are justified by Tonelli/Fubini under σ–finiteness. For F–valued expressions we apply Tonelli to the norm (reduce to a nonnegative scalar integrand), and domination comes from the mixed–norm integrability $\|L\|_{L^q_y(L^p_x)} < \infty$.

Definition 4.2 Let (X, Σ, m) be σ-finite. For a measurable $H : X \times X \to \mathbb{R}$ and $1 \le p < \infty$,

$$\|H(\cdot, y)\|_{L^p_x} := \Big(\int_X |H(x,y)|^p \, dm(x) \Big)^{1/p} \quad \text{for a.e. } y \in X,$$

and for $1 \le q < \infty$,

$$\|h\|_{L^q_y} := \Big(\int_X |h(y)|^q \, dm(y) \Big)^{1/q}.$$

We also write

$$\|H\|_{L^q_y(L^p_x)} := \big\| \, \|H(\cdot, y)\|_{L^p_x} \big\|_{L^q_y} = \Big(\int_X \Big(\int_X |H(x,y)|^p \, dm(x) \Big)^{q/p} dm(y) \Big)^{1/q}.$$

Remark 4.3 ($L^p \to L^p$ under a mixed–norm Lipschitz bound) Let $1 < p < \infty$ and let q be its Hölder conjugate. Assume that for a.e. $(x,y) \in X \times X$ and all $r, s \in \mathbb{R}$,

$$\|\varphi_x(r,y) - \varphi_x(s,y)\|_F \le L(x,y) \, |r - s|,$$

where $L : X \times X \to [0, \infty)$ is measurable. Set

$$\ell(y) := \|L(\cdot, y)\|_{L^p_x} = \Big(\int_X |L(x,y)|^p \, dm(x) \Big)^{1/p} \quad \text{for a.e. } y \in X,$$

and suppose $\ell \in L^q_y$, i.e. $\|\ell\|_{L^q_y} = \|L\|_{L^q_y(L^p_x)} < \infty$. Define the Urysohn-type operator

$$(Tf)(x) := \int_X \varphi_x(f(y), y)\, dm(y) \qquad (f \in L^p(m; S)).$$

Then $T : L^p(m; S) \to L^p(m; F)$ is Lipschitz with

$$\|Tf - Tg\|_{L^p_x(F)} \;\le\; \|L\|_{L^q_y(L^p_x)}\, \|f - g\|_{L^p_y(S)} \qquad (f, g \in L^p(m; S)).$$

In particular, if $\|L\|_{L^q_y(L^p_x)} \le 1$, then T is nonexpansive on $L^p(m; F)$.

Proof (sketch) For $h = f - g$,

$$\|Tf(x) - Tg(x)\|_F \le \int_X L(x, y)\, |h(y)|\, dm(y).$$

Taking the L^p-norm in x and applying Minkowski's integral inequality gives

$$\|Tf - Tg\|_{L^p_x} \le \int_X \|L(\cdot, y)\|_{L^p_x}\, |h(y)|\, dm(y) = \int_X \ell(y)\, |h(y)|\, dm(y).$$

Hölder in y with exponents (q, p) yields

$$\|Tf - Tg\|_{L^p_x} \le \|\ell\|_{L^q_y}\, \|h\|_{L^p_y}.$$

Equivalently, the operator Lipschitz constant is $\|L\|_{L^q_y(L^p_x)}$, which coincides with the modulus furnished by the parametric extension of Theorem 3.2 (parameter y). $\square$

Remark 4.4 (Endpoint cases $p = 1$ and $p = \infty$) The arguments below use Hölder with a finite conjugate exponent, hence $1 < p < \infty$. At the endpoints one needs different (often stronger) hypotheses. For $p = 1$,

$$\|Tf - Tg\|_{L^1_x} \le \Big(\operatorname*{ess\,sup}_{y \in X} \int_X L(x, y)\, \mathrm{d}m(x) \Big)\, \|f - g\|_{L^1_y},$$

i.e. the constant is $\|L\|_{L^\infty_y(L^1_x)}$. For $p = \infty$,

$$\|Tf - Tg\|_{L^\infty_x} \le \Big(\operatorname*{ess\,sup}_{x \in X} \int_X L(x, y)\, \mathrm{d}m(y) \Big)\, \|f - g\|_{L^\infty_y},$$

i.e. the constant is $\|L\|_{L^1_y(L^\infty_x)}$. More refined weak-type/Schur-type assumptions are possible, but we do not pursue endpoints here.

Remark 4.5 By Theorem 2.3, both the Nemytskii and the Urysohn instances admit RN-type kernels. For the Nemytskii case, $(Tf)(x) = \varphi(f(x), x)$ with φ as obtained in Sect. 2 and 3. For the Urysohn operator, write $\varphi_x(r, \cdot)$ so that $(Tf)(x) = \int_X \varphi_x\big(f(y), y\big)\, dm(y)$; this is precisely the parametric form used below.

Example 4.1 (Nemytskii operator) By Theorem 3.2, the Radon—Nikodym kernel φ satisfies $Tf = \int_X \varphi(f(y), y)\, dm(y)$. All nonexpansive/contractive bounds then follow directly from (2) under Assumption 4.1.

Assume that for m-a.e. $y \in X$ the map $r \mapsto \varphi(r, y)$ is Lipschitz on every ball $K_\alpha \subset S$ with constant $L(y)$, and that $L \in L^q(X)$ with $\|L\|_{L^q(X)} \leq 1$ (where $1/p + 1/q = 1$). Suppose also $\varphi(0, y) = 0$ a.e. Then the global integral operator

$$Tf \; = \; \int_X \varphi\big(f(y), y\big)\, dm(y)$$

is well-defined on $L^p(X, S)$ and satisfies

$$\|Tf - Tg\| \; \leq \; \int_X L(y)\, \|f(y) - g(y)\|\, dm(y) \; \leq \; \|L\|_{L^q(X)}\, \|f - g\|_{L^p(X,S)}.$$

Hence T is *nonexpansive*; if $\|L\|_{L^q(X)} < 1$, then T is a *strict contraction*.

Proof (*sketch*) This is exactly (2) using Hölder's inequality and $\varphi(0, y) = 0$. □

Example 4.2 (Urysohn operator) Let $1 < p < \infty$ and let q be its Hölder conjugate. Suppose $K : X \times X \times \mathbb{R} \to \mathbb{R}$ is measurable in (x, y) and for all $r, s \in \mathbb{R}$,

$$|K(x, y, r) - K(x, y, s)| \leq L(x, y)\, |r - s| \quad \text{for a.e. } (x, y) \in X \times X, \tag{5}$$

with a measurable $L \geq 0$. Set, for a.e. $y \in X$,

$$\ell(y) := \|L(\cdot, y)\|_{L^p_x} = \Big(\int_X |L(x, y)|^p\, dm(x) \Big)^{1/p},$$

and assume $\ell \in L^q_y$ with $L_0 := \|\ell\|_{L^q_y} = \|L\|_{L^q_y(L^p_x)} < \infty$. Define

$$(Tf)(x) := \int_X K(x, y, f(y))\, dm(y)$$

whenever the integral exists. Then for all $f, g \in L^p(X)$,

$$\|Tf - Tg\|_{L^p_x} \leq L_0\, \|f - g\|_{L^p_y}.$$

In particular, if $L_0 \leq 1$ then T is nonexpansive on $L^p(X)$; if $L_0 < 1$, then T is a strict contraction.

Proof Let $h := f - g$. By (5), for a.e. x,

$$|Tf(x) - Tg(x)| = \Big| \int_X \big(K(x, y, f(y)) - K(x, y, g(y))\big)\, dm(y) \Big| \leq \int_X L(x, y)\, |h(y)|\, dm(y).$$

Taking the L^p-norm in x and using Minkowski's integral inequality,

$$\|Tf - Tg\|_{L^p_x} \leq \int_X \|L(\cdot, y)\|_{L^p_x}\, |h(y)|\, dm(y) = \int_X \ell(y)\, |h(y)|\, dm(y).$$

By Tonelli, $y \mapsto \|L(\cdot, y)\|_{L^p_x}$ is measurable (since $(x,y) \mapsto |L(x,y)|^p$ is), so the right-hand side is well defined and finite whenever $\|L\|_{L^q_y(L^p_x)} < \infty$. Applying Hölder in the y-variable with exponents (q,p) yields

$$\|Tf - Tg\|_{L^p_x} \le \|\ell\|_{L^q_y} \|h\|_{L^p_y} = L_0 \|f - g\|_{L^p_y}.$$

This proves the claim. □

Remark 4.6 (Explicit link to Theorem 3.2) For each fixed y, the map $r \mapsto K(x,y,r)$ is Lipschitz in r with pointwise modulus $L(\cdot, y)$ in x. Interpreting y as a parameter, the parametric form of Theorem 3.2 identifies the Lipschitz modulus at the operator level with

$$\|\ell\|_{L^q_y} = \left\| \, \|L(\cdot, y)\|_{L^p_x} \right\|_{L^q_y},$$

which is exactly the mixed norm $\|L\|_{L^q_y(L^p_x)}$ appearing above. Hence the bound obtained directly by Minkowski's and Hölder's inequalities is the operator-level Lipschitz constant predicted by the parametric extension of Theorem 3.2. See also classical approaches to nonlinear equations of the Urysohn type [8, 9, 26].

Lemma 4.1 (Consistency of kernels in Examples 1.1–1.2) *Let μ be the operator-valued measure from Example 1.1 or Example 1.2. Then μ satisfies the assumptions of Theorem 3.2, and the Carathéodory kernel φ obtained there agrees a.e. with the kernel used in the example (respectively $\varphi(r,x)$ in Example 1.1 and $\varphi(r,x) = \int_X K(x,y,r)\,dm(y)$ in Example 1.2).*

Proof sketch This follows from the reverse direction (Theorem 3.3) together with the a.e. uniqueness of the Radon—Nikodym kernel in Theorem 3.2. □

Remark 4.7 In Sect. 1 Examples 1.1 and 1.2 stand on their own: the operator-valued measures μ defined there, and the corresponding integrals $\int f\,d\mu$, already yield the classical Nemytskii and Urysohn operators without invoking Sects. 2–3. The conclusions obtained from the theorems in Sect. 3 (Lemma 4.1) justify the representation of these operators in the form with the Caratheodory kernel $\varphi(r,x)$. Only at this stage do we impose the *Lipschitz assumption* on $r \mapsto \varphi(r, \cdot)$ (Assumption 4.1), which yields the nonexpansive/contractive bounds in L^p and, in turn, the standard fixed-point schemes (Halpern, Picard).

Remark 4.8 In Example 4.1 the kernel arises from Theorem 3.2. In Example 4.2, $K(x,y,r) = \varphi_x(r,y)$ corresponds to the family (μ_x) of Sect. 3. Thus both examples are direct corollaries of the Radon-Nikodym representation with the stated Lipschitz conditions.

Corollary 4.1 (Picard, strict case) *If* $\sup_{x \in X} \left(\int_X L(x,y)^p \, dm(y) \right)^{1/p} < 1$ *and* $F = S$, *then* $T : L^p(X,S) \to L^p(X,S)$ *is a strict contraction and the Picard iteration* $f_{n+1} = Tf_n$ *converges in norm to the unique fixed point* $f^* \in L^p(X,S)$.

5 Conclusions

1. *Representation and global operator.* Under semivariation bounds and absolute continuity $\mu \ll m$, the operator-valued measure admits a Radon–Nikodým–type

representation by a (locally uniformly continuous) Carathéodory kernel $\varphi : S \times X \to F$:

$$\mu(E)r = \int_E \varphi(r,x)\,\mathrm{d}m(x) \quad (E \in \mathcal{R},\ r \in S),$$

hence, for admissible f,

$$\int_E f\,\mathrm{d}\mu = \begin{cases} (P)\int_E \varphi\big(f(x),x\big)\,\mathrm{d}m(x) & \text{(Pettis)}, \\ \int_E \varphi\big(f(x),x\big)\,\mathrm{d}m(x) & \text{(Bochner)}. \end{cases}$$

In particular,

$$Tf := \int_X f\,\mathrm{d}\mu = \int_X \varphi\big(f(x),x\big)\,\mathrm{d}m(x) \in F,$$

which is consistent with the setwise construction and stable under standard approximations.

2. *Mixed–norm Lipschitz modulus on L^p.* If for a.e. (x,y) and all $r,s \in \mathbb{R}$

   $$\|\varphi_x(r,y) - \varphi_x(s,y)\|_F \le L(x,y)\,|r-s|$$

   with measurable $L \ge 0$, and if $1 < p < \infty$ with $1/p + 1/q = 1$, then

   $$\|Tf - Tg\|_{L^p_x(F)} \le \|L\|_{L^q_y(L^p_x)}\,\|f-g\|_{L^p_y(S)} \qquad (f,g \in L^p),$$

   so T is nonexpansive when $\|L\|_{L^q_y(L^p_x)} \le 1$ and a strict contraction when < 1. This operator constant agrees with the one predicted by the *parametric extension of Theorem 3.2* (parameter y).
3. *Iterative consequences and the role of Vitali.* The mixed–norm control supplies verifiable hypotheses for the convergence of standard fixed-point schemes (Krasnoselskii–Mann, Halpern). We also make explicit why Vitali-type arguments alone do not yield norm convergence in this setting and how the representation furnishes the regularity needed to recover it.
4. *Classical integral operators as special cases.* Nemytskii and Urysohn operators fit the framework. For Nemytskii, $(Tf)(x) = \varphi(f(x),x)$ with the same kernel φ; for Urysohn, $(Tf)(x) = \int_X K\big(x,y,f(y)\big)\,\mathrm{d}m(y)$ and the parametric kernel $\varphi_x(r,y) = K(x,y,r)$ yields the same mixed–norm Lipschitz modulus $\|L\|_{L^q_y(L^p_x)}$.
5. *Outlook.* Natural directions include: (i) extensions to Orlicz/variable–exponent settings via growth encoded in φ; (ii) refined demiclosedness and Opial–type principles under representation–level moduli; (iii) stability for composed/perturbed kernels and measure-driven evolutions. These would further strengthen the links between RN-type representations and nonexpansive–type operator theory and algorithms.

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References

1. Bartle, R.G.: A general bilinear vector integral. Stud. Math. **15**, 337–352 (1956)
2. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces, 2nd edn. CMS Books in Mathematics. Springer, Cham (2017)
3. Bessaga, C., Pełczyński, A.: On bases and unconditional convergence of series in Banach spaces. Stud. Math. **17**, 151–164 (1958)
4. Bogachev, V.I.: Measure Theory, vol. I. Springer, Berlin (2007)
5. Borwein, J., Reich, S., Shafrir, I.: Krasnoselski-Mann interations in normed spaces. Can. Math. Bull. **35**, 21–28 (1992)
6. Browder, F.E.: Nonexpansive nonlinear operators in a Banach space. Proc. Natl. Acad. Sci. USA **54**(4), 1041–1044 (1965)
7. Candeloro, D., Volčič, A.: Radon–Nikodým theorems. In: Pap, E. (ed.) Handbook of Measure Theory, vol. I, pp. 249–294. Elsevier, Amsterdam (2002)
8. Corduneanu, C.: Integral Equations and Applications. Cambridge University Press, Cambridge (1991)
9. Deimling, K.: Nonlinear Functional Analysis. Springer, Berlin (1985)
10. Diestel, J., Uhl, J.J.: Vector Measures. Mathematical Surveys., vol. 15. Am. Math. Soc., Providence (1977)
11. Dinculeanu, N.: Vector Measures. Pergamon, Oxford (1967)
12. Dinculeanu, N.: Vector Integration and Stochastic Integration in Banach Spaces. Wiley–Interscience, New York (2000)
13. Dobrakov, I.: On integration in Banach spaces I. Czechoslov. Math. J. **20**, 511–536 (1970)
14. Dobrakov, I., Dobrakov, J.: Daniell type extensions of l^1-gauges and integrals. Math. Slovaca **47**(3), 267–281 (1997)
15. Dobrakov, I., Panchapagesan, T.: A generalized Pettis measurability criterion and integration of vector functions. Stud. Math. **164**, 205–229 (2004)
16. Dunford, N., Schwartz, J.T.: Linear Operators, Part I. Interscience, New York (1958)
17. Goebel, K., Kirk, W.A.: Topics in Metric Fixed Point Theory. Cambridge Studies in Advanced Mathematics, vol. 28. Cambridge University Press, Cambridge (1990)
18. Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings. Dekker, New York (1984)
19. Halpern, B.: Fixed points of nonexpanding maps. Bull. Am. Math. Soc. **73**, 957–961 (1967)
20. Hildebrandt, T.H.: Integration in abstract spaces. Bull. Am. Math. Soc. **59**, 111–139 (1953)
21. Jefferies, B.: Bilinear integration and applications to operator and scattering theory. Rocky Mt. J. Math. **46**, 189–225 (2016)
22. Jefferies, B.: Singular Bilinear Integrations. World Scientific, Singapore (2017)
23. Khamsi, M.A., Kozłowski, W.M.: Fixed Point Theory in Modular Function Spaces. Springer, Heidelberg (2015)
24. Kozłowski, W.M.: Modular Function Spaces. Monographs and Textbooks in Pure and Applied Mathematics, vol. 122. Dekker, New York (1988)
25. Krasnoselskii, M.A.: Two remarks on the method of successive approximations. Usp. Mat. Nauk **10**, 123–127 (1955) (in Russian)
26. Krasnoselskii, M.A.: Topological Methods in the Theory of Nonlinear Integral Equations. Macmillan Co., New York (1964)
27. Mann, W.R.: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506–510 (1953)
28. Maynard, H.B.: A general Radon–Nikodym theorem. In: Vector and Operator Valued Measures and Applications, pp. 233–256. Academic Press, New York (1973)
29. Mizel, V.J., Sundaresan, K.: Representation of vector valued nonlinear functions. Trans. Am. Math. Soc. **159**, 111–127 (1971)
30. Moedomo, S., Uhl, J.J.: Radon–Nikodym theorems for the Bochner and Pettis integrals. Pac. J. Math. **38**, 531–536 (1971)
31. Musiał, K.: A Radon–Nikodym theorem for the Bartle–Dunford–Schwartz integral. Atti Semin. Mat. Fis. Univ. Modena **41** (1993)
32. Musiał, K.: Pettis integral. In: Handbook of Measure Theory, vol. I, pp. 531–586. Elsevier, Amsterdam (2002)
33. Nowak, M.: Operators on the space of bounded strongly measurable functions. J. Math. Anal. Appl. **388**, 393–403 (2012)
34. Nowak, M.: Operator measures and integration operators. Indag. Math. **24**, 279–290 (2013)
35. Nowak, M.: Bochner representable operators on Banach function spaces. Positivity **22**, 1303–1309 (2018)

36. Opial, Z.: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. **73**, 591–597 (1967)
37. Panchapagesan, T.V.: On the distinguishing features of the Dobrakov integral. Divulg. Mat. **3**, 79–114 (1995)
38. Reich, S.: The fixed point property for nonexpansive mappings I. Am. Math. Mon. **83**, 266–268 (1976)
39. Reich, S.: Weak convergence theorems for nonexpansive mappings. J. Math. Anal. Appl. **67**, 274–276 (1979)
40. Reich, S.: The fixed point property for nonexpansive mappings II. Am. Math. Mon. **87**, 292–294 (1980)
41. Reich, S.: Strong convergence theorem for resolvents of accretive operators in Banach spaces. J. Math. Anal. Appl. **75**, 287–292 (1980)
42. Reich, S.: Approximating fixed points of nonexpansive mappings. Panam. Math. J. **4**(2), 23–28 (1994)
43. Reich, S., Sabach, S.: A strong convergence theorem for Halpern-type iterations in Banach spaces. J. Fixed Point Theory Appl. **22**(4), Paper No. 73, 17 (2020)
44. Reich, S., Sabach, S.: Two strong convergence theorems for a proximal method in reflexive Banach spaces. Numer. Funct. Anal. Optim. **41**(10), 1206–1223 (2020)
45. Reich, S., Zaslavski, A.J.: Convergence of Krasnoselskii-Mann iterations of nonexpansive operators. Math. Comput. Model. **32**, 1423–1431 (2000)
46. Rieffel, M.A.: The Radon–Nikodym theorem for the Bochner integral. Trans. Am. Math. Soc. **131**, 446–487 (1968)
47. Szczypiński, T.: On the representation of nonlinear operators defined on spaces of measurable functions. In: Colloquia Mathematica Societatis János Bolyai; Function, Series, Operators, vol. II, vol. 35, pp. 1183–1188. North-Holland, Amsterdam (1983)
48. Szczypiński, T.: Non-linear operator valued measures and integration. Ann. Soc. Math. Pol., 1 Comment. Math. **27**(2), 313–333 (1988)

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